Assignment 8

This homework is due *Monday*, November 21.

There are total 45 points in this assignment. 35 points is considered 100%. If you go over 35 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations (unless stated otherwise in the problem). Bare answers will not earn you much.

This assignment covers sections 4.6–4.8 in O'Neill.

(1) (4.6.2) Let $\alpha : [-1, 1] \to \mathbb{R}^2$ be the curve segment given by $\alpha(t) = (t, t^2)$.
- (a) [2pt] If $\varphi = v^2 du + 2uvdv$, compute $\int_\alpha \varphi$.
- (b) [3pt] Find a function f such that $df = \varphi$. Compute $\int_\beta \varphi$, where β is a straight line joining $\alpha(-1)$ and $\alpha(1)$. Compare to (a).

(2) (4.6.3) Let φ be a 1-form on a surface M. Show:
- (a) [3pt] If φ is closed, then $\int_{\partial \mathbf{x}} \varphi = 0$ for every 2-segment $\mathbf{x}$ in M.
- (b) [3pt] If φ is exact,

$$\int_\alpha \varphi = \sum_i \int_{\alpha_i} \varphi = 0,$$

for any closed, piecewise smooth curve whose smooth segments are $\alpha_1, \alpha_2, \ldots, \alpha_k$. (In particular, α_k end point is the start point of α_1.)

(3) (4.6.9) Let $\mathbf{x}$ be the usual parametrization of the torus T:

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u).$$

For integers m, n let α be the closed curve

$$\alpha(t) = \mathbf{x}(mt, nt) \quad (0 \le t \le 2\pi).$$

Find:
- (a) [3pt] $\int_\alpha \xi$, where ξ is the 1-form such that $\xi(\mathbf{x}_u) = 1$ and $\xi(\mathbf{x}_v) = 0$.
- (b) [3pt] $\int_\alpha \eta$, where η is the 1-form such that $\eta(\mathbf{x}_u) = 0$ and $\eta(\mathbf{x}_v) = 1$.

Comment. For an arbitrary closed curve γ on the torus, $\int_\gamma \xi/(2\pi)$ counts the number of turns around the torus the curve makes in direction of the parallels, and $\int_\gamma \eta/(2\pi)$ does the same for the meridians.

(4) (Simplified 4.7.9) Let φ be a *closed* 1-form and α be a closed curve.
- (a) [2pt] Show that if either φ is exact or α is homotopic to a constant, then $\int_\alpha \varphi = 0$.
- (b) [3pt] Deduce that on the torus T the meridians and parallels are not homotopic to constants, and the closed 1-forms ξ and η are not exact. (In particular, show that they are closed.)

— *see next page* —

(5) [3pt] (4.7.3) Let $F : M \to N$ be a regular mapping. Prove that if N is orientable, then M is orientable. (*Hint:* Use the Pullback.)

(6) [3pt] (4.8.1) Prove that a surface M is nonorientable if there is a smoothly closed curve $\alpha : [a, b] \to M$ (it means it is closed and $\alpha'(a) = \alpha'(b)$) and a tangent vector field Y on α such that
(i) Y and α' are linearly independent at every point, and
(ii) $Y(b) = -Y(a)$.
(*Hint:* Assume M is orientable, and drag nonvanishing 2-form η computed on Y and α' along α. Keep track of sign of η.)

(7) (4.8.11) (*Plane with two origins*) Let Z consist of all ordered pairs of real numbers and one additional point $\mathbf{0}^*$. Let $\mathbf{x}$ and $\mathbf{y}$ be the functions from $\mathbb{R}^2$ to Z such that

$$\mathbf{x}(u, v) = \mathbf{y}(u, v) = (u, v) \quad \text{if} \quad (u, v) \neq (0, 0),$$

but

$$\mathbf{x}(0, 0) = \mathbf{0} = (0, 0) \quad \text{and} \quad \mathbf{y}(0, 0) = \mathbf{0}^*.$$

(a) [3pt] Show that the abstract patches $\mathbf{x}$ and $\mathbf{y}$ constitute a patch collection that satisfies the first two conditions in the definition of (abstract) surface, but not the Hausdorff axiom.
Without Hausdorff axiom, strange things happen. For example, prove the following.

(b) [3pt] A convergent sequence in Z can have two limits.

(c) [3pt] The function $F : Z \to Z$ that reverses $\mathbf{0}$ and $\mathbf{0}^*$, leaving all other points fixed, is a differentiable mapping.

(8) (4.8.12)

(a) [4pt] Given a one-to-one function H from a manifold M onto an arbitrary set A, prove there is a unique way to make A a manifold so that H becomes a diffeomorphism. (*Hint:* Diffeomorphisms carry patches to patches. So since you need to produce a collection of patches in A, show that they must be images of patches in M under H.)

(b) [4pt] In each of the following cases, find natural choices of H and M that make the set a manifold.

(i) The set of all 2×2 real symmetric matrices.

(ii) The set of all circles in $\mathbb{R}^2$.

(iii) The set of all great (largest) circles on a sphere Σ.

(iv) The set of all finite closed intervals of nonzero length in $\mathbb{R}$.